Multi-objective minimum time optimal control for low-thrust trajectory design

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I.

Proposition 1.1: Under Assumption 3.2, any trajectory $\mathbf{r}$ with terminal time t_f reconstructed from f is guaranteed to be Lipschitz continuous, with Lipschitz constant $L_{t_f} := 1 + t_f L_f e^{t_f L_f}$.

Proof: Let $(r_0, t_f), (\hat{r}_0, \hat{t}_f) \in \mathbb{R}^7 \times [0, \infty)$ be two initial states and terminal times, then by Carathéodory's existence theorem, the following relation holds for arbitrary input policies $\mathbf{u}, \hat{\mathbf{u}} \in \mathcal{U}_{ad}$ and $t \in [0, 1]$:

$$\begin{aligned} &|\mathbf{r}(t) - \hat{\mathbf{r}}(t)| \\ &\leq |r_0 - \hat{r}_0| + t_f \int_0^t \left| \widetilde{f}(\mathbf{r}(s), \mathbf{u}(s)) - \widetilde{f}(\hat{\mathbf{r}}(s), \hat{\mathbf{u}}(s)) \right| ds \\ &\leq |r_0 - \hat{r}_0| + t_f L_f \int_0^t |\mathbf{r}(s) - \hat{\mathbf{r}}(s)| \, ds. \end{aligned}$$

Using the Bellman-Gronwall Lemma [1] it then follows that

$$\begin{aligned} |\mathbf{r}(t) - \hat{\mathbf{r}}(t)| &\leq |r_0 - \hat{r}_0| + \int_0^t |r_0 - \hat{r}_0| \, t_f L_f e^{t \cdot t_f L_f} ds \\ &\leq |r_0 - \hat{r}_0| \left(1 + t_f L_f e^{t_f L_f}\right) = L_{t_f} |r_0 - \hat{r}_0| . \end{aligned}$$

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II.

Proposition 2.1: The value function ω is Lipschitz continuous.

Proof: Let us fix $(r_0, z, t_f), (\hat{r}_0, \hat{z}, \hat{t}_f) \in \mathbb{R}^7 \times \mathbb{R}^2 \times [0, \infty)$, $\kappa \in [0, 1]$ and let $\epsilon > 0$. We choose $\hat{\mathbf{r}} \in \Pi_{\hat{r}_0, \hat{t}_f}$ such that

$$\begin{aligned} &\omega(\kappa, \hat{r}_0, \hat{z}, \hat{t}_f) \\ &\geq \bigvee_i J_i(\hat{t}_f, \hat{\mathbf{r}}(1)) - \hat{z}_i \bigvee \nu(\hat{\mathbf{r}}(1)) \bigvee \max_{s \in [\kappa, 1]} g(\hat{\mathbf{r}}(s)) - \epsilon. \end{aligned}$$

By definition of ω, for any $\mathbf{r} \in \Pi_{\hat{r}_0, \hat{t}_f}$, this yields the following relation

$$\begin{aligned} &\omega(\kappa, r_0, z, t_f) - \omega(\kappa, \hat{r}_0, \hat{z}, \hat{t}_f) \\ &\quad \leq \bigvee_i J_i(t_f, \mathbf{r}(1)) - z_i \bigvee \nu(\mathbf{r}(1)) \bigvee \max_{s \in [\kappa, 1]} g(\mathbf{r}(s)) \\ &- \bigvee_i J_i(\hat{t}_f, \hat{\mathbf{r}}(1)) - \hat{z}_i \bigvee \nu(\hat{\mathbf{r}}(1)) \bigvee \max_{s \in [\kappa, 1]} g(\hat{\mathbf{r}}(s)) + \epsilon. \end{aligned}$$

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Let $\kappa_0 \in [\kappa, 1]$ be such that $g(\mathbf{r}(\kappa_0)) = \max_{s \in [\kappa, 1]} g(\mathbf{r}(s))$. Then subsequently $-\max_{s \in [\kappa, 1]} g(\hat{\mathbf{r}}(s)) \leq g(\hat{\mathbf{r}}(\kappa_0))$ and

$$\begin{aligned} &\omega(\kappa, r_0, z, t_f) - \omega(\kappa, \hat{r}_0, \hat{z}, \hat{t}_f) \\ &\quad \leq \bigvee_i J_i(t_f, \mathbf{r}(1)) - z_i \bigvee \nu(\mathbf{r}(1)) \bigvee g(\mathbf{r}(\kappa_0)) \\ &\quad - \bigvee_i J_i(\hat{t}_f, \hat{\mathbf{r}}(1)) - \hat{z}_i \bigvee \nu(\hat{\mathbf{r}}(1)) \bigvee g(\hat{\mathbf{r}}(\kappa_0)) + \epsilon. \end{aligned}$$

Using Proposition 1.1, we define $L_r := L_{t_f} \bigvee L_{\hat{t}_f}$ and show that in every case, there exists a Lipschitz constant.

Case 1: $g(\mathbf{r}(\kappa_0)) \geq \omega(\kappa, r_0, z, t_f)$

$$\begin{aligned} &\omega(\kappa, r_0, z, t_f) - \omega(\kappa, \hat{r}_0, \hat{z}, \hat{t}_f) \\ &\leq g(\mathbf{r}(\kappa_0)) - \bigvee_i J_i(\hat{t}_f, \hat{\mathbf{r}}(1)) - \hat{z}_i \bigvee \nu(\hat{\mathbf{r}}(1)) \bigvee g(\hat{\mathbf{r}}(\kappa_0)) + \epsilon \\ &\quad \leq g(\mathbf{r}(\kappa_0)) - g(\hat{\mathbf{r}}(\kappa_0)) + \epsilon \leq L_g L_r |r_0 - \hat{r}_0| + \epsilon \end{aligned}$$

For the following cases the argumentation remains the same as in Case 1, and we simply state the final inequality.

Case 2: $\nu(\mathbf{r}(1)) \geq \omega(\kappa, r_0, z, t_f)$

$$\omega(\kappa, r_0, z, t_f) - \omega(\kappa, \hat{r}_0, \hat{z}, \hat{t}_f) \leq L_\nu L_r |r_0 - \hat{r}_0| + \epsilon$$

Case 3: $J_2(t_f, \mathbf{r}(1)) - z_2 \geq \omega(\kappa, r_0, z, t_f)$

$$\omega(\kappa, r_0, z, t_f) - \omega(\kappa, \hat{r}_0, \hat{z}, \hat{t}_f) \leq \left|t_f - \hat{t}_f\right| + |z_2 - \hat{z}_2| + \epsilon$$

Case 4: $J_1(t_f, \mathbf{r}(1)) - z_1 \geq \omega(\kappa, r_0, z, t_f)$

$$\omega(\kappa, r_0, z, t_f) - \omega(\kappa, \hat{r}_0, \hat{z}, \hat{t}_f) \leq |r_7 - \hat{r_7}| + |z_1 - \hat{z}_1| + \epsilon$$

Thus in every case there exists a set of constants C_r, C_z and C_{t_f} such that

$$\begin{aligned} &\omega(\kappa, r_0, z, t_f) - \omega(\kappa, \hat{r}_0, \hat{z}, \hat{t}_f) \\ &\quad \leq C_r |r_0 - \hat{r}_0| + C_z |z - \hat{z}| + C_{t_f} \left|t_f - \hat{t}_f\right| + \epsilon \end{aligned}$$

The same argument conducted with (r_0, z, t_f) and $(\hat{r}_0, \hat{z}, \hat{t}_f)$ reversed establishes that

$$\begin{aligned} &\omega(\kappa, \hat{r}_0, \hat{z}, \hat{t}_f) - \omega(\kappa, r_0, z, t_f) \\ &\quad \leq C_r |r_0 - \hat{r}_0| + C_z |z - \hat{z}| + C_{t_f} \left|t_f - \hat{t}_f\right| + \epsilon \end{aligned}$$

Since ϵ is arbitrary, we conclude that

$$\begin{aligned} &\left|\omega(\kappa, \hat{r}_0, \hat{z}, \hat{t}_f) - \omega(\kappa, r_0, z, t_f)\right| \\ &\quad \leq C_r |r_0 - \hat{r}_0| + C_z |z - \hat{z}| + C_{t_f} \left|t_f - \hat{t}_f\right| \end{aligned}$$

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References

[1] S. Sastry, *Nonlinear systems : analysis, stability, and control*, ser. Interdisciplinary applied mathematics ; v. 10. New York ; London: Springer, 1999.